2024 HKDSE Mathematics Extended Part (Module 1) Examination

Suggested Solutions

1a	$0.3+a+0.1+0.2+0.2=1$ $a=0.2$ $\mathrm{E}(X)=0\times0.3+3\times0.2+6\times0.1+b\times0.2+15\times0.2$ $=0.2b+4.2$ $\mathrm{E}(X^2)=0^2\times0.3+3^2\times0.2+6^2\times0.1+b^2\times0.2+15^2\times0.2$ $=0.2b^2+50.4$ $\mathrm{Var}(5X)=739$ $5^2\,\mathrm{Var}(X)=739$ $\mathrm{Var}(X)=29.56$ $\mathrm{E}(X^2)-[\mathrm{E}(X)]^2=29.56$ $(0.2b^2+50.4)-(0.2b+4.2)^2=29.56$ $0.16b^2-1.68b+3.2=0$ $2b^2-21b+40=0$ $(2b-5)(b-8)=0$ $b=\frac{5}{2}$ (rej.) or 8 $b=8$
bi	$\mathrm{P}(C)=0.2+0.1$ $=0.3$ $\mathrm{P}(D)=0.1+0.2+0.2$ $=0.5$ $\mathrm{P}(C\cap D)=0.1$

	$$\begin{aligned}\mathrm{P}(C)\mathrm{P}(D) &= 0.3\times 0.5\\ &= 0.15\\ &\neq 0.1\\ &= \mathrm{P}(C\cap D)\end{aligned}$$ Therefore, *C* and *D* are not independent.
ii	0.7
2a	$$\begin{aligned}\text{The required probability} &= \frac{\left(\frac{3}{20}\right)}{1-\frac{3}{5}}\\ &= \frac{3}{8}\end{aligned}$$
b	Let *p* be the probability that a randomly selected member is a female. $$\begin{aligned}(1-p)\times\left(1-\frac{4}{9}\right)+\frac{3}{20} &= 1-\frac{3}{5}\\ p &= \frac{11}{20}\end{aligned}$$ $$\begin{aligned}\text{The required probability} &= \frac{11}{20}\times\left[1-\frac{\left(\frac{3}{20}\right)}{\left(\frac{11}{20}\right)}\right]\\ &= \frac{2}{5}\end{aligned}$$

3a	$$\left[C_1^{20}(1-p)^{20-1}p^1\right]:\left[C_3^{20}(1-p)^{20-3}p^3\right]=49:57$$ $$\frac{20(1-p)^{19}p}{1140(1-p)^{17}p^3}=\frac{49}{57}$$ $$(1-p)^2=49p^2$$ $$1-p=7p\quad(\because 0<p<1)$$ $$p=\frac{1}{8}$$
b	$$1-\left(1-\frac{1}{8}\right)^k>0.85$$ $$0.875^k<0.15$$ $$k>\log_{0.875}0.15\quad(\because 0.875^x \text{ is strictly decreasing})$$ $$k>14.20729573$$ Therefore, the least value of k is 15.
4a	Let z be a number such that $\mathrm{P}(-z\le Z\le z)=\frac{\beta}{100}$. $$2\times z\times\frac{1.75}{\sqrt{81}}=0.7$$ $$z=1.8$$ $$\beta=100P(-1.8\le Z\le 1.8)$$ $$\approx 100\times 2\times 0.4641$$ $$=92.8200\quad(\text{corr. to 4 d.p.})$$
b	The new mean $=\frac{81\times 13-36\times 12.5}{81-36}$ $=13.4$ hours The new sample standard deviation $$=\sqrt{\frac{1}{81-36-1}\left(\left\{\left[1.75^2\times(81-1)+81\times 13^2\right]-\left[2^2\times(36-1)+36\times 12.5^2\right]\right\}-(81-36)\times 13.4^2\right)}$$ $$=\sqrt{\frac{111}{55}}$$ $$=\frac{\sqrt{6105}}{55}\text{ hours}$$

5a	$\frac{2}{e^{nx}} = 2e^{-nx}$ $= 2\left[1 + \frac{-nx}{1!} + \frac{(-nx)^2}{2!} + \frac{(-nx)^3}{3!} + \ldots\right]$ $= 2 - 2nx + n^2x^2 - \frac{n^3}{3}x^3 + \ldots$
b	The general term of $(1+4x)^m = C_r^m (4x)^r$ $= C_r^m 4^r x^r$ The coefficient of $x = 24$ $C_1^m 4^1 + (-2n) = 24$ $4m - 2n = 24$ $n = 2m - 12$ The coefficient of $x^2 = 980$ $C_2^m 4^2 + n^2 = 980$ $16\left[\frac{m(m-1)}{2}\right] + (2m-12)^2 = 980$ $3m^2 - 14m - 209 = 0$ $(3m+19)(m-11) = 0$ $m = -\frac{19}{3}$ (rej.) or 11 $m = 11$ $n = 2(11) - 12$ $= 10$ The coefficient of $x^3 = C_3^{11} 4^3 + \left(-\frac{10^3}{3}\right)$ $= \frac{30680}{3}$

6ai	$e^u = \left(x^2+x+e\right)^{2x+1}$ $u = (2x+1)\ln\left(x^2+x+e\right)$ Hence, $\mathrm{p}(x) = 2x+1$ and $\mathrm{q}(x) = x^2+x+e$.
ii	$\frac{\mathrm{d}}{\mathrm{d}x}e^u = e^u\frac{\mathrm{d}u}{\mathrm{d}x}$ $= \left(x^2+x+e\right)^{2x+1}\left[2\ln\left(x^2+x+e\right)+\frac{2x+1}{x^2+x+e}\times(2x+1)\right]$ $= \left(x^2+x+e\right)^{2x+1}\left[2\ln\left(x^2+x+e\right)+\frac{(2x+1)^2}{x^2+x+e}\right]$
b	When $x = 0$, $y = \left(0^2+0+e\right)^{2(0)+1}$ $= e$ The slope of the tangent $= \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{x=0}$ $= \left.\frac{\mathrm{d}}{\mathrm{d}x}e^u\right\|_{x=0}$ $= \left(0^2+0+e\right)^{2(0)+1}\left[2\ln\left(0^2+0+e\right)+\frac{(2\times 0+1)^2}{0^2+0+e}\right]$ $= e\left(2+\frac{1}{e}\right)$ $= 2e+1$ Therefore, the equation of the tangent to Γ at H is $y = (2e+1)x+e$, i.e. $(2e+1)-y+e = 0$.

7	Let y cm, z cm and A cm^2 be the length, diagonal and the area of the rectangle respectively. Note that any interior angle of a rectangle is $90°$ (def. of rectangle). $$x^2+y^2=z^2 \quad (\text{Pyth. Thm.})$$ $$\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2+y^2\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(z^2\right)$$ $$2x\frac{\mathrm{d}x}{\mathrm{d}t}+2y\frac{\mathrm{d}y}{\mathrm{d}t}=0$$ $$x\frac{\mathrm{d}x}{\mathrm{d}t}+y\frac{\mathrm{d}y}{\mathrm{d}t}=0$$ $$A=xy$$ $$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}(xy)$$ $$\frac{\mathrm{d}A}{\mathrm{d}t}=y\frac{\mathrm{d}x}{\mathrm{d}t}+x\frac{\mathrm{d}y}{\mathrm{d}t}$$ When $x=7$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=-0.5$, $$7^2+y^2=15^2+20^2$$ $$y=24 \quad (\because y>0)$$ $$7(-0.5)+24\frac{\mathrm{d}y}{\mathrm{d}t}=0$$ $$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{7}{48}$$ The required rate of change $=\frac{\mathrm{d}A}{\mathrm{d}t}$ $$=24(-0.5)+7\left(\frac{7}{48}\right)$$ $$=-\frac{527}{48} \text{ cm s}^{-1}$$

8a	$\int_0^{0.5} e^{\frac{-x^2}{2}}\,\mathrm{d}x = \sqrt{2\pi}\int_0^{0.5}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}\,\mathrm{d}x$ $\approx \sqrt{2\pi}\times \mathrm{P}(0\le z\le 0.5)$ $\approx \sqrt{2\pi}\times 0.1915$ ≈ 0.480019314 $= 0.4800 \quad$ (corr. to 4 d.p.)
b	Put $y=0$, $(2x-1)e^{\frac{-x^2}{2}} = 0$ $2x-1=0 \quad \left(\because e^{\frac{-x^2}{2}} > 0 \text{ for all } x\ge 0\right)$ $x=\frac{1}{2}$ When $0<x<\frac{1}{2}$, $y=(2x-1)e^{\frac{-x^2}{2}}<0$. The required area $= -\int_0^{0.5} y\,\mathrm{d}x$ $= -\int_0^{0.5}(2x-1)e^{\frac{-x^2}{2}}\,\mathrm{d}x$ $= -\int_0^{0.5}e^{\frac{-x^2}{2}}\,\mathrm{d}(x^2)+\int_0^{0.5}e^{\frac{-x^2}{2}}\,\mathrm{d}x$ $\approx \left[e^{-\frac{x^2}{2}}\right]_0^{0.5} + 0.480019314$ $= 0.6132 \quad$ (corr. to 4 d.p.)

9a	$\mathrm{P}\left(\frac{(\mu-1.5)-\mu}{\sigma}<Z<\frac{(\mu+1.5)-\mu}{\sigma}\right)=0.7888$ $\mathrm{P}\left(\frac{-1.5}{\sigma}<Z<\frac{1.5}{\sigma}\right)=0.7888$ $\frac{1.5}{\sigma}\approx 1.25$ $\sigma=1.2000$ (corr. to 4 d.p.) $\mathrm{P}\left(Z>\frac{5.7-\mu}{\sigma}\right)=0.3085$ $\frac{5.7-\mu}{1.2000}\approx 0.5$ $\sigma=5.1000$ (corr. to 4 d.p.)
b	The mean weight of these pumpkins follows $\mathrm{N}\left(\mu,\frac{\sigma^2}{16}\right)$. The required probability $=\mathrm{P}\left(Z\le\frac{5.4-\mu}{\sqrt{\frac{\sigma^2}{16}}}\right)$ $\approx\mathrm{P}\left(Z\le\frac{5.4-5.1}{\left(\frac{1.2}{4}\right)}\right)$ $\approx\mathrm{P}(Z\le 1)$ $\approx 0.5+0.3413$ $=0.8413$ (corr. to 4 d.p.)
ci	$\mathrm{P}(\text{Grade C})=\mathrm{P}\left(Z\le\frac{3.6-\mu}{\sigma}\right)$ $\approx\mathrm{P}\left(Z\le\frac{3.6-5.1}{1.2}\right)$ $\approx\mathrm{P}(Z\le -1.25)$ $\approx 0.5-0.3944$ ≈ 0.1056 $\mathrm{P}(\text{Grade B})\approx 1-0.3085-0.1056$ ≈ 0.5859 $\mathrm{P}(\text{Grade A})=0.3085$

	The expected price $\approx (50\times 0.1056+80\times 0.5859+100\times 0.3085)\times 8$ $= 664.0160$ kg (corr. to 4 d.p.)
ii	The required probability $\approx \frac{8!}{2!5!}(0.1056)^2(0.5859)^5(0.3085)+\frac{8!}{6!}(0.1056)(0.5859)^6(0.3085)$ $+\frac{8!}{5!2!}(0.1056)(0.5859)^5(0.3085)^2+8(0.5859)^7(0.3085)$ $+C_2^8(0.5859)^6(0.3085)^2+C_3^8(0.5859)^5(0.3085)^3$ $=0.5101$ (corr. to 4 d.p.)
10a	The required probability $=e^{-1.6}\left(1+\frac{1.6^1}{1!}+\frac{1.6^2}{2!}\right)$ $=\frac{3.88}{e^{1.6}}$ $=0.7834$ (corr. to 4 d.p.)
b	The reuqired probability $=\left(\frac{3.88}{e^{1.6}}\right)^7$ ≈ 0.181018883 $=0.1810$ (corr. to 4 d.p.)
c	The required probability $=\frac{C_2^7(e^{-1.6}\times 1)^2\left(e^{-1.6}\times\frac{1.6^2}{2!}\right)^5+\frac{7!}{2!4!}(e^{-1.6}\times 1)\left(e^{-1.6}\times\frac{1.6^1}{1!}\right)^2\left(e^{-1.6}\times\frac{1.6^2}{2!}\right)^4+C_3^7\left(e^{-1.6}\times\frac{1.6^1}{1!}\right)^4\left(e^{-1.6}\times\frac{1.6^2}{2!}\right)^3}{\left(\frac{3.88}{e^{1.6}}\right)^7}$ $=0.0963$ (corr. to 4 d.p.)
d	The required probability $=\frac{\left(\frac{3.88}{e^{1.6}}\right)^7-\left[e^{-1.6}\left(\frac{1.6^1}{1!}+\frac{1.6^2}{2!}\right)\right]^7-7(e^{-1.6}\times 1)\left[e^{-1.6}\left(\frac{1.6^1}{1!}+\frac{1.6^2}{2!}\right)\right]^6}{1-\left[1-e^{-1.6}(1)\right]^7-7\left[1-e^{-1.6}(1)\right]^{7-1}\left[e^{-1.6}(1)\right]}$ $=0.2425$ (corr. to 4 d.p.)

11a	$\ln\left(\frac{P}{-t^2+10t+8}\right)=\ln\frac{a\left(-t^2+10t+8\right)e^{bt}}{-t^2+10t+8}$ $=\ln ae^{bt}$ $=bt+\ln a$
b	Since the intercept on the horizontal axis of the graph of the linear function is 2.5, $0=b(2.5)+\ln a$ $\ln a=-2.5b$ Since the graph of the linear function passes through the point with coordinates $(3,-0.1)$, $-0.1=b(3)+\ln a$ $-0.1=3b+(-2.5b)$ $b=-0.2$ $\ln a=-2.5(-0.2)$ $\ln a=0.5$ $a=e^{0.5}$ $a=\sqrt{e}$ Therefore, $\begin{cases} a=\sqrt{e} \\ b=-0.2 \end{cases}$.
c	Let $\mathrm{f}(t)=\sqrt{e}\left(-t^2+10t+8\right)e^{-0.2t}$. The accumulative rainfall $=\int_0^4 P\,\mathrm{d}t$ $=\int_0^4 \sqrt{e}\left(-t^2+10t+8\right)e^{-0.2t}\,\mathrm{d}t$ $\approx\frac{4-0}{2(4)}\{\mathrm{f}(0)+\mathrm{f}(4)+2[\mathrm{f}(1)+\mathrm{f}(2)+\mathrm{f}(3)]\}$ ≈ 94.15996350 $=94.1600$ (corr. to 4 d.p.)

di	$\int Q\,\mathrm{d}t = \int \frac{16(2t+5)e^{0.4t}}{4te^{0.4t}+3}\,\mathrm{d}t$ $= 16\int \frac{2t+5}{4t+3e^{-0.4t}}\,\mathrm{d}t$ $= 4\int \frac{2(4t+3e^{-0.4t})+5(4-1.2e^{-0.4t})}{4t+3e^{-0.4t}}\,\mathrm{d}t$ $= 8\int \mathrm{d}t + 20\int \frac{\mathrm{d}(4t+3e^{-0.4t})}{4t+3e^{-0.4t}}$ $= 8t + 20\ln\left|4t+3e^{-0.4t}\right| + C$ $= 20\ln\left[e^{\frac{8}{20}}(4t+3e^{-0.4t})\right] + C \quad (\because t>0 \text{ and } e^{-0.4t}>0)$ $= 20\ln(4te^{0.4t}+3) + C$
ii	$\mathrm{f}'(t) = \sqrt{e}\left[(-2t+10)e^{-0.2t} - 0.2(-t^2+10t+8)e^{-0.2t}\right]$ $= 0.2\sqrt{e}(t^2-20t+42)e^{-0.2t}$ For any $0<t<4$, $\mathrm{f}''(t) = 0.2\sqrt{e}\left[(2t-20)e^{-0.2t} - 0.2(t^2-20t+42)e^{-0.2t}\right]$ $= -0.04(t^2-30t+142)e^{-0.2t}$ $= 0.04\left[-(t-15)^2+83\right]e^{-0.2t}$ $\le 0.04\left[-(4-15)^2+83\right]e^{-0.2t} \quad (\because 0<t<4)$ $= -1.52e^{-0.2t}$ $< 0 \quad (\because e^{-0.2t}>0 \text{ for any real number } t)$ Therefore, $\mathrm{f}(t)$ is concave downwards on $[0,4]$. Hence, the estimate in (c) is an underestimate. The sum concerned $= \int_0^4 P\,\mathrm{d}t + \int_0^4 Q\,\mathrm{d}t$ $= \int_0^4 P\,\mathrm{d}t + \left[20\ln(4te^{0.4t}+3)\right]_0^4$ $= \int_0^4 P\,\mathrm{d}t + 20\ln\frac{16e^{1.6}+3}{3}$ $> 94.15996350 + 20\ln\frac{16e^{1.6}+3}{3}$ ≈ 160.3826253 mm > 160 mm Hence, the claim is agreed with.

12a	Method using differentiation

<u>Method using differentiation</u>

$$\frac{d^2R}{dt^2}=\frac{d}{dt}\left(\frac{2e^t-5}{2e^t-5e^{0.5t}+5}+2\right)$$

$$=\frac{2e^t}{2e^t-5e^{0.5t}+5}-\frac{(2e^t-5)(2e^t-2.5e^{0.5t})}{(2e^t-5e^{0.5t}+5)^2}$$

$$=\frac{2e^t(2e^t-5e^{0.5t}+5)-e^{0.5t}(2e^t-5)(2e^{0.5t}-2.5)}{(2e^t-5e^{0.5t}+5)^2}$$

$$=\frac{e^{0.5t}(-5e^t+20e^{0.5t}-12.5)}{(2e^t-5e^{0.5t}+5)^2}$$

$$=-\frac{5e^{0.5t}(2e^t-8e^{0.5t}+5)}{2(2e^t-5e^{0.5t}+5)^2}$$

Put $\frac{d^2R}{dt^2}=0$,

$$-\frac{5e^{0.5t}(2e^t-8e^{0.5t}+5)}{2(2e^t-5e^{0.5t}+5)^2}=0$$

$$2e^t-8e^{0.5t}+5=0 \quad (\because e^{0.5t}>0 \text{ for any real number } t)$$

$$e^{0.5t}=\frac{-(-8)\pm\sqrt{(-8)^2-4(2)(5)}}{2(2)}$$

$$e^{0.5t}=\frac{4-\sqrt{6}}{2} \quad \text{or} \quad \frac{4+\sqrt{6}}{2}$$

$$0.5t=\ln\frac{4-\sqrt{6}}{2} \quad \text{or} \quad \ln\frac{4+\sqrt{6}}{2}$$

$$t=2\ln\frac{4-\sqrt{6}}{2} \ (\text{rej.}) \quad \text{or} \quad 2\ln\frac{4+\sqrt{6}}{2}$$

$$t=2\ln\frac{4+\sqrt{6}}{2}$$

t	$0<t<2\ln\frac{4+\sqrt{6}}{2}$	$t>2\ln\frac{4+\sqrt{6}}{2}$
$\frac{d^2R}{dt^2}$	+	−

Therefore, the rate of change of the total revenue attains a maximum when $t=2\ln\frac{4+\sqrt{6}}{2}$.

	The greatest rate of change $=\left.\dfrac{\mathrm{d}R}{\mathrm{d}t}\right\|_{t=2\ln\frac{4+\sqrt{6}}{2}}$ $=\dfrac{2\left(\dfrac{4+\sqrt{6}}{2}\right)^2-5}{2\left(\dfrac{4+\sqrt{6}}{2}\right)^2-5\left(\dfrac{4+\sqrt{6}}{2}\right)+5}+2$ $=\dfrac{12+8\sqrt{6}}{12+3\sqrt{6}}+2$ $=\dfrac{4\left(3+2\sqrt{6}\right)}{3\left(4+\sqrt{6}\right)}\times\dfrac{4-\sqrt{6}}{4-\sqrt{6}}+2$ $=\dfrac{4\left(5\sqrt{6}\right)}{3\left(4^2-6\right)}+2$ $=\dfrac{2}{3}\sqrt{6}+2$ $<\dfrac{2}{3}(3)+2$ $=4$ thousand per month Hence, the claim is not agreed with. <u>Method not using differentiation</u> Suppose that the rate of change of the total revenue of the shop can exceed 4 thousand dollars per month. $\dfrac{2e^{0.5t}-5e^{-0.5t}}{2e^{0.5t}+5e^{-0.5t}-5}+2>4$ $\dfrac{2e^{t}-5}{2e^{t}-5e^{0.5t}+5}>2$ $2e^{t}-5>4e^{t}-10e^{0.5t}+10 \quad \left(\begin{array}{l}\because 2e^{t}-5e^{0.5t}+5\\ =2\left(e^{0.5t}-\dfrac{5}{4}\right)^2+\dfrac{15}{8}\\ \geq 0+\dfrac{15}{8}\\ >0 \text{ for any real number } t\end{array}\right)$ $2\left(e^{0.5t}\right)^2-10e^{0.5t}+15<0$

Consider the equation $2x^2 - 10x + 15 = 0$, where $x = e^{0.5t}$.

$$\begin{aligned}\Delta &= (-10)^2 - 4(2)(15) \\ &= -20 \\ &< 0\end{aligned}$$

Since the coefficient of x^2 is $2 > 0$, we know that $2x^2 - 10x + 15 > 0$ for any real number *x*, which is a contradiction.
Hence, the greatest rate of change concerned does not exceed 4 thousand dollars per month.

bi

$$\begin{aligned}\text{The total profit} &= \int_0^{12} \frac{\mathrm{d}P}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_0^{12} \left[\frac{\mathrm{d}R}{\mathrm{d}t} - 10(0.8)^{2t+3}\right]\mathrm{d}t \\
&= \int_0^{12} \left[\left(\frac{2e^{0.5t} - 5e^{-0.5t}}{2e^{0.5t} + 5e^{-0.5t} - 5} + 2\right) - 10(0.8)^{2t+3}\right]\mathrm{d}t \\
&= \int_0^{12} \frac{2e^{0.5t} - 5e^{-0.5t}}{2e^{0.5t} + 5e^{-0.5t} - 5}\,\mathrm{d}t + \int_0^{12} \left(2 - 5.12 \times 0.64^t\right)\mathrm{d}t \\
&= 2\int_0^{12} \frac{\mathrm{d}\left(2e^{0.5t} + 5e^{-0.5t} - 5\right)}{2e^{0.5t} + 5e^{-0.5t} - 5} + \left[2t - \frac{5.12 \times 0.64^t}{\ln 0.64}\right]_0^{12} \\
&= 2\left[\ln\left|2e^{0.5t} + 5e^{-0.5t} - 5\right|\right]_0^{12} + 24 - \frac{5.12\left(0.64^{12} - 1\right)}{\ln 0.64} \\
&= 2\ln\frac{2e^6 + 5e^{-6} - 5}{5} + 24 + \frac{5.12}{\ln 0.64} - \frac{5.12 \times 0.64^{12}}{\ln 0.64} \\
&\approx 24.56934013 \text{ thousand dollars} \\
&= 24.5693 \text{ thousand dollars} \quad (\text{corr. to 4 d.p.})\end{aligned}$$

ii

$$\begin{aligned}\lim_{t\to+\infty} \frac{\mathrm{d}P}{\mathrm{d}t} &= \lim_{t\to+\infty}\left[\frac{\mathrm{d}R}{\mathrm{d}t} + 10(0.8)^{2t+3}\right] \\
&= \lim_{t\to+\infty}\left(\frac{2e^{0.5t} - 5e^{-0.5t}}{2e^{0.5t} + 5e^{-0.5t} - 5} + 2 + 5.12 \times 0.64^t\right) \\
&= \lim_{t\to+\infty}\left(\frac{2 - 5e^{-t}}{2 + 5e^{-t} - 5e^{-0.5t}} + 2 + 5.12 \times 0.64^t\right) \\
&= \frac{2 - 5(0)}{2 + 5(0) - 5(0)} + 2 + 5.12 \times 0 \\
&= 3\end{aligned}$$

Therefore, the rate of change of the total profit of the shop after a very long time is 3 thousand dollars per month.